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Exact Solutions of Some Nonlinear Evolution Equations Using Symbolic Computations

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Abstract—In this paper, we present a solution methodology that utilizes symbolic computations to obtain analytic solutions of some nonlinear evolution equations by balancing the nonlinear and the dispersive effects. The solution method is demonstrated by obtaining solutions to Burgers' equation, the nonlinear heat equation, the modified KdV equation, and the Kuramoto-Sivashinsky equation.

Keywords—Nonlinear partial differential equations, Symbolic computation, Evolution equations, Mapping nonlinear equations, Exact solutions, KdV equation, Burgers' equation, Kuramoto-Sivashinsky equation.

1. INTRODUCTION

In various fields of science and engineering, nonlinear evolution equations, as well as their analytic solutions, are of fundamental importance. Analytic solutions of such equations are either not available or obtained using transformations based on, for instance, one parameter Lie-Backlund groups analysis [1–3], invariance group analysis [4], and Lie infinitesimal criterion [5]. The use of symbolic computation as a useful tool in the investigation of nonlinear evolution equations has been recently demonstrated by finding an exact solution to the generalized breaking soliton equation [6] and the generalized shallow water wave equation [7]. A recent investigation utilized a second-order splitting method to find an approximate numerical solution to the Kuramoto-Sivashinsky (KS) equation [8].

In this work, we use a symbolic computation based approach to establish transformations appropriate for reducing some nonlinear evolution equations into the well-known linear heat equation. The method is also used to reduce the modified KdV equation and the KS equation into a system of functions that are homogeneous in the partial derivatives which leads to exact solutions. The method can be outlined in the following algorithm.

(1) Assume a solution composed of linear combination of a function $f(w)$ and some of its partial derivatives in the form

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$$u = \frac{\partial^{m+n} f(w)}{\partial x^m \partial t^n} + \sum_{\substack{i=0,\\ j=0}}^{\substack{j=n-1,\\ i=m-1}} a_{i+j} \frac{\partial^{i+j} f(w)}{\partial x^i \partial t^j} + b, \tag{1.1}$$

where $w = w(x,t)$ is a quasi-solution of the nonlinear equation. Both f and w are to be determined.

(2) An estimation of the value of m and n is obtained by balancing the convective and the advective terms in the nonlinear equation, that is by requiring that the nonlinear term and the highest order partial derivative are partially balanced.

(3) An expression for u is suggested from Step 2. The substitution of such expression into the nonlinear equation results in evaluating the transformation needed and the reduction of the nonlinear equation to a linear equation is established.

In the following sections, we demonstrate the convective-advective effects balance approach, outlined above, by finding transformations which map the solution of the Burgers' equation and the nonlinear heat equation onto the solution of the linear heat equation. The relation between the obtained transformations and known transformations obtained elsewhere is discussed for each case. As an example of utilizing the mapping criterion in obtaining analytic solutions of nonlinear equations, the initial value problem of Burgers' equation is discussed. The effectiveness of the method and the usefulness of symbolic computations are demonstrated by finding an exact solution to the modified KdV equation.

2. BURGERS' EQUATION

The nonlinear partial differential equation

$$vu_{xx} - uu_x - u_t = 0, \tag{2.1}$$

known as Burgers' equation, represents a simple model in the theory of turbulence [9] and is used to describe the propagation of one-dimensional acoustic signals of moderate amplitudes [10,11] as well as waves of small amplitude [12]. It has been shown independently by Cole [2] and Hopf [3] that Burgers' equation may be mapped to the linear heat equation using the Backlund transformation.

In order to find a transformation using the approach outlined in the previous section, we start by assuming u as given in equation (1.1). Balancing the convective and advective terms, i.e., requiring that the highest partial derivative and the nonlinear terms be partially balanced, gives

$$vf^{(m+n+2)} w_x^{m+2} w_t^n = f^{(2m+2n+1)} w_x^{2m+1} w_t^{2n},$$

leading to $m = 1$ and $n = 0$. Thus, u can be assumed in the form

$$u = \frac{\partial f}{\partial x} + b. \tag{2.2}$$

Substituting for u in Burgers' equation (2.1) yields

$$\begin{aligned} &-f''(w)w_t w_x - bf''(w)w_x^2 - f'(w)f''(w)w_x^3 + vf^{(3)}(w)w_x^3 - f'(w)w_{xt} \\ &-bf'(w)w_{xx} - f'(w)^2 w_x w_{xx} + 3vf''(w)w_x w_{xx} + vf'(w)w_{xxx} = 0, \end{aligned} \tag{2.3}$$

where the subscripts denote partial derivatives and the prime denotes ordinary derivative. Setting to zero the coefficients of w_x^3 yields the ordinary differential equation

$$-f'(w)f''(w) + vf^{(3)}(w) = 0, \tag{2.4}$$

which is satisfied by

$$f(w) = -2v\ln(w). \tag{2.5}$$

Accordingly, we use the substitution $f'(w)^2 = 2vf''(w)$ in equation (2.3) to get

$$-f''(w)w_xw_t - bf''(w)w_x^2 - f'(w)w_{xt} - bf'(w)w_{xx} + vf''(w)w_xw_{xx} + vf'(w)w_{xxx} = 0. \tag{2.6}$$

Setting the coefficient of f'' equal to zero yields

$$-w_tw_x - bw_x^2 + vw_xw_{xx} = 0. \tag{2.7}$$

Setting the coefficient of f' equal to zero yields

$$-w_{xt} - bw_{xx} + vw_{xxx} = 0. \tag{2.8}$$

Equations (2.7) and (2.8) are equivalent to the linear equation

$$w_t + bw_x - vw_{xx} = 0, \tag{2.9}$$

and the solution of which gives the unknown function $w(x,t)$. If we set $b = 0$ in equation (2.9), we get the linear heat equation

$$w_t - vw_{xx} = 0, \tag{2.10}$$

and the transformation is established from equations (2.2) and (2.5) as

$$u = -2v\frac{w_x}{w}. \tag{2.11}$$

One can show that Backlund transformation is obtainable from equation (2.11) by noticing that

$$w_x = \frac{-1}{2v}uw \tag{2.12}$$

and

$$w_t = \frac{u^2w}{4v} - \frac{wu_x}{2}. \tag{2.13}$$

Equations (2.12) and (2.13) give the transformations derived by Cole [2] and Hopf [3].

2.1. Application

We briefly show the application of the transformation obtained in the current work to the solution of the initial value problem for Burgers' equation. The transformation deduced in equation (2.11) indicates that the initial value problem [10,12],

$$\begin{aligned} vu_{xx} - uu_x - u_t &= 0, \qquad t > 0, \\ u &= g(x), \qquad t = 0, \end{aligned} \tag{2.14}$$

corresponds to the initial value problem for the classical heat equation:

$$\begin{aligned} w_t - vw_{xx} &= 0, \qquad t > 0, \\ w &= e^{-(1/2v)\int_0^x g(\eta)\,d\eta}, \qquad t = 0. \end{aligned} \tag{2.15}$$

The solution of this problem is well known and is given by

$$w = \frac{1}{\sqrt{4\pi vt}}\int_{-\infty}^{\infty} e^{\left(-(1/2v)\int_0^\xi g(\eta)\,d\eta + (x-\xi)^2/2t\right)}\,d\xi. \tag{2.16}$$

By the transformation in (2.11), the solution of the original initial value problem (2.14) is then given by

$$u = \frac{\displaystyle\int_{-\infty}^{\infty} ((x-\xi)/t)\, e^{\left(-(1/2v)\int_0^\xi g(\eta)\,d\eta + (x-\xi)^2/2t\right)}\,d\xi}{\displaystyle\int_{-\infty}^{\infty} e^{\left(-(1/2v)\int_0^\xi g(\eta)\,d\eta + (x-\xi)^2/2t\right)}\,d\xi}. \tag{2.17}$$

Special cases of this solution can be found in [2,12].

3. THE NONLINEAR HEAT EQUATION

It was shown [4] that the nonlinear heat equation

$$u_{xx} + \left(\frac{1}{2}\right)(u_x)^2 - u_t = 0 \tag{3.1}$$

admits a Lie point transformation which maps the equation to the linear heat equation. We now obtain an equivalent transformation using the proposed approach. By balancing the convective and advective effects, one finds that $m = n = 0$. We, therefore, assume the solution to be in the form

$$u = f(w),$$

where $w = w(x,t)$. Both f and w are to be determined later. Substituting for u in the nonlinear equation (3.1) yields

$$-f'w_t + \frac{1}{2}f'^2 w_x^2 + f'' w_x^2 + f' w_{xx} = 0. \tag{3.2}$$

Collecting the coefficients of the w_x^2 terms and setting the result equals to zero gives

$$\frac{f'^2}{2} + f'' = 0. \tag{3.3}$$

The solution of this equation is in the form $f(w) = 2\ln(w/2 + h(x,t))$. Setting the coefficient of f' in equation (3.2) equal to zero gives

$$w_t - w_{xx} = 0. \tag{3.4}$$

Thus, we obtain the mapping

$$u = 2\ln\left(2w(x,t) + h(x,t)\right), \tag{3.5}$$

where $h(x,t)$ is an arbitrary function of x and t. It is easy to check that the transformation (3.5) maps equation (3.1) to the linear equation

$$w_t - w_{xx} + 2\left(h_{xx} - h_t\right) = 0. \tag{3.6}$$

Setting $h = 0$, we see that the transformation

$$u = 2\ln\left(2w(x,t)\right) \tag{3.7}$$

defines a solution $u(x,t)$ of the nonlinear equation (3.1) for any solution $w(x,t)$ of the linear heat equation (3.4). Moreover, the inverse of (3.7) is

$$w = 2e^{u/2}, \tag{3.8}$$

which maps the nonlinear heat equation (3.1) to the linear heat equation (3.4). Equations (3.7) and (3.8) are equivalent to the transformation obtained in [4] using the invariance analysis.

The linear heat equation (3.4) has a fundamental solution in the form

$$w(x,t) = \frac{1}{\sqrt{t}} e^{-x^2/4t}\left[c_1 + c_2\frac{x}{t}\right]. \tag{3.9}$$

The solution of the nonlinear heat equation is reconstructed from equation (3.9) through the transformation (3.7). One then gets

$$u(x,t) = \frac{x^2}{2t} + \ln\frac{t^3}{c_1 + c_2 x}. \tag{3.10}$$

The result of (3.10) satisfies equation (3.1).

4. THE MODIFIED KDV EQUATION

The modified KdV equation takes the form

$$u_t + u^2 u_x - u_{xxx} = 0. \tag{4.1}$$

Using (1.1), the nonlinear term is transformed into

$$u^2 u_x = \left(f^{m+n}(w)\right)^2 \left(f^{m+n+1}(w)\right) w_x^{3m+1} w_t^{3n} + \text{all terms of lower degree.} \tag{4.2}$$

The highest order partial derivative term is transformed into

$$u_{xxx} = f^{m+n+3}(w) w_x^{m+3} w_t^n + \text{all terms of lower degree.} \tag{4.3}$$

Balancing the partial derivatives in (4.2) and (4.3) results in $m = 1$ and $n = 0$. Therefore, we take

$$u = a\frac{\partial}{\partial x} f(w) + b. \tag{4.4}$$

Evaluating each term of (4.1) using (4.4), we obtain

$$u_t = af'' w_t w_x + af' w_{tx}, \tag{4.5}$$

$$\begin{aligned} u^2 u_x &= ab^2 f'' w_x^2 + 2a^2 b f' f'' w_x^3 + a^3 f'^2 f'' w_x^4 \\ &\quad + ab^2 f' w_{xx} + 2a^2 b f'^2 w_x w_{xx} + a^3 f'^3 w_x^2 w_{xx}, \end{aligned} \tag{4.6}$$

$$u_{xxx} = af^{(4)} w_x^4 + 6af^{(3)} w_x^2 w_{xx} + 3af'' w_{xx}^2 + 4af'' w_x w_{xxx} + af' w_{xxxx}, \tag{4.7}$$

where the argument of $f(w)$ is dropped for clarity. Inserting the above results into equation (4.1) yields

$$\begin{aligned} &af'' w_t w_x + ab^2 f'' w_x^2 + 2a^2 b f' f'' w_x^3 + a^3 f'^2 f'' w_x^4 \\ &-af^{(4)} w_x^4 + af' w_{tx} + ab^2 f' w_{xx} + 2a^2 b f'^2 w_x w_{xx} + a^3 f'^3 w_x^2 w_{xx} \\ &-6af^{(3)} w_x^2 w_{xx} - 3af'' w_{xx}^2 - 4af'' w_x w_{xxx} - af' w_{xxxx} = 0. \end{aligned} \tag{4.8}$$

Collecting the coefficient of w_x^4 and setting the result to zero, we obtain the following ordinary differential equation for f:

$$a^2 f'^2 f'' - f^{(4)} = 0. \tag{4.9}$$

We assume the solution of (4.9) in the form

$$f = \sigma \ln w. \tag{4.10}$$

Equation (4.10) satisfies (4.9) provided that

$$\sigma = \pm\frac{\sqrt{6}}{a}. \tag{4.11}$$

Turning to (4.8) and setting the coefficient of $(w_x)^4$ to zero, it reduces to

$$\begin{aligned} &af'' w_t w_x + ab^2 f'' w_x^2 + 2a^2 b f' f'' w_x^3 + af' w_{tx} + ab^2 f' w_{xx} + 2a^2 b f'^2 w_x w_{xx} \\ &+a^3 f'^3 w_x^2 w_{xx} - 6af^{(3)} w_x^2 w_{xx} - 3af'' w_{xx}^2 - 4af'' w_x w_{xxx} - af' w_{xxxx} = 0. \end{aligned} \tag{4.12}$$

Now making the substitutions

$$\begin{aligned} f' f'' &= -\frac{\sigma}{2} f^{(3)}, \\ f'^2 &= -\sigma f'', \\ f'^3 &= \frac{\sigma^2}{2} f^{(3)}, \end{aligned} \tag{4.13}$$

into (4.12), yields

$$\begin{aligned}&af''w_tw_x + ab^2f''w_x^2 - a^2bcf^{(3)}w_x^3 + af'w_{tx} + ab^2f'w_{xx} - 2a^2bcf''w_xw_{xx}\\&-6af^{(3)}w_x^2w_{xx} + \frac{1}{2}a^3c^2f^{(3)}w_x^2w_{xx} - 3af''w_{xx}^2 - 4af''w_xw_{xxx} - af'w_{xxxx} = 0.\end{aligned} \tag{4.14}$$

Collecting the coefficients of $f^{(3)}$, f'' and f', then setting the results, respectively, to zero we obtain

$$abcw_x^3 + \left(6 - \frac{1}{2}a^2c^2\right) w_x^2w_{xx}^3 = 0, \tag{4.15}$$

$$w_tw_x + b^2w_x^2 - 2abcw_xw_{xx} - 3w_{xx}^2 - 4w_xw_{xxx} = 0, \tag{4.16}$$

$$w_{tx} + b^2w_{xx} - w_{xxxx} = 0. \tag{4.17}$$

Equations (4.15) and (4.16), are homogeneous functions of the third and second partial derivatives of $w(x,t)$, respectively, and equation (4.17) is a linear one. Therefore, we can assume a solution to the system in the form

$$w(x,t) = 1 + e^{\alpha x+\beta t+\gamma}, \tag{4.18}$$

where α, β, and γ are to be determined later.

Substituting (4.18) in into (4.15)–(4.17) and simplifying, we obtain, respectively,

$$\left(6 - \frac{1}{2}a^2c^2\right) \alpha + 2abc = 0, \tag{4.19}$$

$$7\alpha^3 + 2abc\alpha^2 - b^2\alpha - \beta = 0, \tag{4.20}$$

$$\alpha^3 - b^2\alpha - \beta = 0. \tag{4.21}$$

Solving (4.19) for α and substituting for c using (4.11) gives

$$\alpha = \mp\sqrt{\frac{2}{3}}b. \tag{4.22}$$

Solving (4.21) for β and substituting for α gives

$$\beta = \pm\sqrt{\frac{2}{27}}b. \tag{4.23}$$

Equations (4.22) and (4.23) satisfy (4.20) for arbitrary values of a and b.

Finally, substituting (4.10) into (4.4) and using (4.18) along with (4.22) and (4.23), we obtain the following pair of exact solution:

$$u = \pm\sqrt{6}\alpha\frac{e^{\alpha x+\beta t+\gamma}}{1 + e^{\alpha x+\beta t+\gamma}} + b.$$

On using the trigonometric identity $e^u/(1+e^u) = (1/2)(\tanh(u/2)+1)$, we put the exact solution in the form

$$u = -b\left(\tanh\frac{\phi_\pm}{2} + 1\right) + b, \tag{4.24}$$

where

$$\phi_\pm = \alpha x + \beta t + \gamma$$

with arbitrary values of b and γ.

5. THE KURAMOTO-SIVASHINSKY EQUATION

The Kuramoto-Sivashinsky equation takes the form

$$u_t + uu_x + u_{xx} + vu_{xxxx} = 0. \tag{5.1}$$

This equation arises in a variety of applications among which are the modeling of reaction-diffusion systems, viscous flow problems, flame propagation, and the study of turbulence phenomena in chemistry. The KS equation is considered to be a prototype of a system with "self-generated" chaos in the large class of generalized Burgers' equations. Numerical experiments related to the KS equation have been performed by several authors [13–15], for instance. In a more recent study, a second-order splitting method is applied to the KS equation and then an orthogonal cubic spline collocation procedure is employed to the approximate resulting system [8]. In this section, we use the methodology outlined in Section 1 to find an analytic solution to the KS equation. Using (1.1), the nonlinear term is transformed into

$$uu_x = \left(f^{m+n}(w)\right)\left(f^{m+n+1}(w)\right) w_x^{2m+1} w_t^{2n} + \text{all terms of lower degree,} \tag{5.2}$$

and the highest order partial derivative term is transformed into

$$u_{xxxx} = f^{m+n+4}(w) w_x^{m+4} w_t^{n} + \text{all terms of lower degree.} \tag{5.3}$$

Balancing the partial derivatives in (5.2) and (5.3) results in $m = 3$ and $n = 0$. Therefore, we take

$$u = a\frac{\partial^3}{\partial x^3} f(w) + b\frac{\partial^2}{\partial x^2} f(w) + c\frac{\partial}{\partial x} f(w) + d. \tag{5.4}$$

Evaluating each term in (5.1) using (5.4), we obtain

$$\begin{aligned}
u_t + uu_x + u_{xx} + vu_{xxxx} = {} & cf''(w)w_t w_x + cdf''(w)w_x^2 + bf^{(3)}(w)w_t w_x^2 \\
& + c^2 f'(w)f''(w)w_x^3 + cf^{(3)}(w)w_x^3 + bdf^{(3)}(w)w_x^3 \\
& + af^{(4)}(w)w_t w_x^3 + bcf''(w)^2 w_x^4 + bcf'(w)f^{(3)}(w)w_x^4 \\
& + bf^{(4)}(w)w_x^4 + adf^{(4)}(w)w_x^4 + b^2 f''(w)f^{(3)}(w)w_x^5 \\
& + acf''(w)f^{(3)}(w)w_x^5 + acf'(w)f^{(4)}(w)w_x^5 + af^{(5)}(w)w_x^5 \\
& + cvf^{(5)}(w)w_x^5 + abf^{(3)}(w)^2 w_x^6 + abf''(w)f^{(4)}(w)w_x^6 \\
& + bvf^{(6)}(w)w_x^6 + a^2 f^{(3)}(w)f^{(4)}(w)w_x^7 + avf^{(7)}(w)w_x^7 \\
& + cf'(w)w_{tx} + 2bf''(w)w_x w_{tx} + 3af^{(3)}(w)w_x^2 w_{tx} \\
& + cdf'(w)w_{xx} + bf''(w)w_t w_{xx} + c^2 f'(w)^2 w_x w_{xx} \\
& + 3cf''(w)w_x w_{xx} + 3bdf''(w)w_x w_{xx} \\
& + 3af^{(3)}(w)w_t w_x w_{xx} + 5bcf'(w)f''(w)w_x^2 w_{xx} \\
& + 6bf^{(3)}(w)w_x^2 w_{xx} + 6adf^{(3)}(w)w_x^2 w_{xx} + 3b^2 f''(w)^2 w_x^3 w_{xx} \\
& + 3acf''(w)^2 w_x^3 w_{xx} + b^2 f'(w)f^{(3)}(w)w_x^3 w_{xx} \\
& + 7acf'(w)f^{(3)}(w)w_x^3 w_{xx} + 10af^{(4)}(w)w_x^3 w_{xx} \\
& + 10cvf^{(4)}(w)w_x^3 w_{xx} + 12abf''(w)f^{(3)}(w)w_x^4 w_{xx} \\
& + abf'(w)f^{(4)}(w)w_x^4 w_{xx} + 15bvf^{(5)}(w)w_x^4 w_{xx} \\
& + 6a^2 f^{(3)}(w)^2 w_x^5 w_{xx} + 3a^2 f''(w)f^{(4)}(w)w_x^5 w_{xx} \\
& + 21avf^{(6)}(w)w_x^5 w_{xx} + 3af''(w)w_{tx}w_{xx} + bcf'(w)^2 w_{xx}^2
\end{aligned} \tag{5.5}$$

$$
\begin{aligned}
&+ 3bf''(w)w_{xx}^2 + 3adf''(w)w_{xx}^2 \\
&+ 3b^2f'(w)f''(w)w_xw_{xx}^2 + 6acf'(w)f''(w)w_xw_{xx}^2 \\
&+ 15af^{(3)}(w)w_xw_{xx}^2 + 15cvf^{(3)}(w)w_xw_{xx}^2 \\
&+ 12abf''(w)^2w_x^2w_{xx}^2 + 6abf'(w)f^{(3)}(w)w_x^2w_{xx}^2 \\
&+ 45bvf^{(4)}(w)w_x^2w_{xx}^2 + 21a^2f''(w)f^{(3)}(w)w_x^3w_{xx}^2 \\
&+ 105avf^{(5)}(w)w_x^3w_{xx}^2 + 3abf'(w)f''(w)w_{xx}^3 \\
&+ 15bvf^{(3)}(w)w_{xx}^3 + 9a^2f''(w)^2w_xw_{xx}^3 \\
&+ 105avf^{(4)}(w)w_xw_{xx}^3 + bf'(w)w_{txx} \\
&+ 3af''(w)w_xw_{txx} + cf'(w)w_{xxx} + bdf'(w)w_{xxx} \\
&+ af''(w)w_tw_{xxx} + bcf'(w)^2w_xw_{xxx} \\
&+ 4bf''(w)w_xw_{xxx} + 4adf''(w)w_xw_{xxx} \\
&+ b^2f'(w)f''(w)w_x^2w_{xxx} + 5acf'(w)f''(w)w_x^2w_{xxx} \\
&+ 10af^{(3)}(w)w_x^2w_{xxx} + 10cvf^{(3)}(w)w_x^2w_{xxx} \\
&+ 4abf''(w)^2w_x^3w_{xxx} + 2abf'(w)f^{(3)}(w)w_x^3w_{xxx} \\
&+ 20bvf^{(4)}(w)w_x^3w_{xxx} + 4a^2f''(w)f^{(3)}(w)w_x^4w_{xxx} \\
&+ a^2f'(w)f^{(4)}(w)w_x^4w_{xxx} + 35avf^{(5)}(w)w_x^4w_{xxx} \\
&+ b^2f'(w)^2w_{xx}w_{xxx} + acf'(w)^2w_{xx}w_{xxx} \\
&+ 10af''(w)w_{xx}w_{xxx} + 10cvf''(w)w_{xx}w_{xxx} \\
&+ 10abf'(w)f''(w)w_xw_{xx}w_{xxx} + 60bvf^{(3)}(w)w_xw_{xx}w_{xxx} \\
&+ 12a^2f''(w)^2w_x^2w_{xx}w_{xxx} + 6a^2f'(w)f^{(3)}(w)w_x^2w_{xx}w_{xxx} \\
&+ 210avf^{(4)}(w)w_x^2w_{xx}w_{xxx} + 3a^2f'(w)f''(w)w_{xx}^2w_{xxx} \\
&+ 105avf^{(3)}(w)w_{xx}^2w_{xxx} + abf'(w)^2w_{xxx}^2 \\
&+ 10bvf''(w)w_{xxx}^2 + 4a^2f'(w)f''(w)w_xw_{xxx}^2 \\
&+ 70avf^{(3)}(w)w_xw_{xxx}^2 + af'(w)w_{txxx} \\
&+ bf'(w)w_{xxxx} + adf'(w)w_{xxxx} \\
&+ acf'(w)^2w_xw_{xxxx} + 5af''(w)w_xw_{xxxx} \\
&+ 5cvf''(w)w_xw_{xxxx} + abf'(w)f''(w)w_x^2w_{xxxx} \\
&+ 15bvf^{(3)}(w)w_x^2w_{xxxx} + a^2f'(w)f^{(3)}(w)w_x^3w_{xxxx} \\
&+ 35avf^{(4)}(w)w_x^3w_{xxxx} + abf'(w)^2w_{xx}w_{xxxx} \\
&+ 15bvf''(w)w_{xx}w_{xxxx} + 3a^2f'(w)f''(w)w_xw_{xx}w_{xxxx} \\
&+ 105avf^{(3)}(w)w_xw_{xx}w_{xxxx} + a^2f'(w)^2w_{xxx}w_{xxxx} \\
&+ 35avf''(w)w_{xxx}w_{xxxx} + af'(w)w_{xxxxx} \\
&+ cvf'(w)w_{xxxxx} + 6bvf''(w)w_xw_{xxxxx} \\
&+ 21avf^{(3)}(w)w_x^2w_{xxxxx} + 21avf''(w)w_{xx}w_{xxxxx} \\
&+ bvf'(w)w_{xxxxxx} + 7avf''(w)w_xw_{xxxxxx} \\
&+ avf'(w)w_{xxxxxxx}.
\end{aligned}
\tag{5.5)(cont.}
$$

Collecting the coefficient of w_x^7 and setting the result to zero, we obtain

$$af^{(3)}(w)f^{(4)}(w) + vf^{(7)}(w) = 0. \tag{5.6}$$

The solution of this ordinary differential equation takes the form

$$f = \mu \ln w, \tag{5.7}$$

which upon substituting into (5.6) specifies the value of μ to be

$$\mu = \frac{60v}{a}.$$

We now make the following substitutions into equation (5.5):

$$\begin{aligned}
f' f^{(2)} &= -\left(\frac{\mu}{2}\right) f^{(3)}, \\
f' f^{(3)} &= -\left(\frac{\mu}{3}\right) f^{(4)}, \\
f' f^{(4)} &= -\left(\frac{\mu}{4}\right) f^{(5)}, \\
f'' f^{(3)} &= -\left(\frac{\mu}{12}\right) f^{(5)}, \\
f'' f^{(4)} &= -\left(\frac{\mu}{20}\right) f^{(6)}, \\
(f')^2 &= -\mu f'', \\
(f'')^2 &= -\left(\frac{\mu}{6}\right) f^{(4)}, \\
(f''')^2 &= -\left(\frac{\mu}{30}\right) f^{(6)},
\end{aligned} \tag{5.8}$$

and simplify to get

$$\begin{aligned}
&-4bvf^{(6)}(w)w_x^6 + \left(aw_x^5 - \frac{5b^2v}{a}w_x^5 - 19cvw_x^5 - 60bvw_x^4w_{xx}\right) f^{(5)}(w) \\
&+ \Bigg(aw_tw_x^3 + bw_x^4 + a\,dw_x^4 - \frac{30bcv}{a}w_x^4 + 10aw_x^3w_{xx} - \frac{50b^2v}{a}w_x^3w_{xx} \\
&-160cvw_x^3w_{xx} - 195bvw_x^2w_{xx}^2 + 15avw_xw_{xx}^3 - 60bvw_x^3w_{xxx} \\
&-30\,avw_x^2w_{xx}w_{xxx} + 15avw_x^3w_{xxxx}\Bigg) f^{(4)}(w) + \Bigg(bw_tw_x^2 + cw_x^3 + b\,dw_x^3 \\
&-\frac{30c^2v}{a}w_x^3 + 3aw_x^2w_{tx} + 3aw_tw_xw_{xx} + \frac{90b^2v}{a}w_xw_{xx}^2 - 165cvw_xw_{xx}^2 \\
&-75bvw_{xx}^3 + 10aw_x^2w_{xxx} - \frac{30b^2v}{a}w_x^2w_{xxx} - 140cvw_x^2w_{xxx} \\
&-240bvw_xw_{xx}w_{xxx} + 15avw_{xx}^2w_{xxx} - 50avw_xw_{xxx}^2 - 15bvw_x^2w_{xxxx} \\
&+15a\,vw_xw_{xx}w_{xxxx} + 21avw_x^2w_{xxxxx}\Bigg) f^{(3)}(w) \\
&+ \Bigg(cw_tw_x + c\,dw_x^2 + 2bw_xw_{tx} + bw_tw_{xx} + 3cw_xw_{xx} + 3b\,dw_xw_{xx} \\
&-\frac{60c^2v}{a}w_xw_{xx} + 3aw_{tx}w_{xx} + 3bw_{xx}^2 + 3a\,dw_{xx}^2 - \frac{60bcv}{a}w_{xx}^2 \\
&+3aw_xw_{txx} + aw_tw_{xxx} + 4bw_xw_{xxx} + 4a\,dw_xw_{xxx} - \frac{60bcv}{a}w_xw_{xxx} \\
&+10aw_{xx}w_{xxx} - \frac{60b^2v}{a}w_{xx}w_{xxx} - 50cvw_{xx}w_{xxx} - 50bvw_{xxx}^2 + 5aw_xw_{xxxx} \\
&-55cvw_xw_{xxxx} - 45bvw_{xx}w_{xxxx} - 25avw_{xxx}w_{xxxx} + 6bvw_xw_{xxxxx} \\
&+21\,avw_{xx}w_{xxxxx} + 7avw_xw_{xxxxxx}\Bigg) f''(w) + (cw_{tx} + c\,dw_{xx} + bw_{txx} + cw_{xxx} \\
&+b\,dw_{xxx} + aw_{txxx} + bw_{xxxx} + a\,dw_{xxxx} + aw_{xxxxx} \\
&+cvw_{xxxxx} + bvw_{xxxxxx} + a\,vw_{xxxxxxx})\, f'(w) = 0.
\end{aligned} \tag{5.9}$$

Setting each of the coefficients of $f^{(6)}$, $f^{(5)}$, $f^{(4)}$, $f^{(3)}$, f'', and f' to zero, we obtain, respectively,

$$-4bvw_x^6 = 0, \tag{5.10}$$

$$\left(a - \frac{5b^2v}{a} - 19cv\right) w_x^5 - 60bvw_x^4 w_{xx} = 0, \tag{5.11}$$

$$\begin{aligned}
\left(b + a\,d - \frac{30bcv}{a}\right) w_x^4 + 15avw_x w_{xx}^3 - 195bvw_x^2 w_{xx}^2 \\
-30avw_x^2 w_{xx} w_{xxx} + aw_t w_x^3 + \left(10a - \frac{50b^2v}{a} - 160cv\right) w_x^3 w_{xx} \\
-60bvw_x^3 w_{xxx} + 15avw_x^3 w_{xxxx} = 0,
\end{aligned} \tag{5.12}$$

$$\begin{aligned}
\left(c + b\,d - \frac{30c^2v}{a}\right) w_x^3 - 75bvw_{xx}^3 + 15avw_{xx}^2 w_{xxx} \\
+\left(15a - \frac{90b^2v}{a} - 165cv\right) w_x w_{xx}^2 - 50avw_x w_{xxx}^2 + 3aw_t w_x w_{xx} \\
-240bvw_x w_{xx} w_{xxx} + 15avw_x w_{xx} w_{xxxx} + bw_t w_x^2 + 3aw_{tx} w_x^2, \\
+\left(6b + 6a\,d - \frac{150bcv}{a}\right) w_x^2 w_{xx} + \left(10a - \frac{30b^2v}{a} - 140cv\right) w_x^2 w_{xxx} \\
-15bvw_x^2 w_{xxxx} + 21avw_x^2 w_{xxxxx} = 0,
\end{aligned} \tag{5.13}$$

$$\begin{aligned}
c\,dw_x^2 + \left(3b + 3a\,d - \frac{60bcv}{a}\right) w_{xx}^2 - 50bvw_{xxx}^2 \\
+aw_t w_{xxx} - 25avw_{xxx} w_{xxxx} \\
+bw_t w_{xx} + 3aw_{tx} w_{xx} + \left(10a - \frac{60b^2v}{a} - 50cv\right) w_{xx} w_{xxx} \\
-45bvw_{xx} w_{xxxx} + 21avw_{xx} w_{xxxxx} \\
+cw_t w_x + 2bw_{tx} w_x + \left(3c + 3b\,d - \frac{60c^2v}{a}\right) w_x w_{xx} \\
+3aw_{txx} w_x + \left(4b + 4a\,d - \frac{60bcv}{a}\right) w_x w_{xxx} + (5a - 55cv)\, w_x w_{xxxx} \\
+6bvw_x w_{xxxxx} + 7avw_x w_{xxxxxx} = 0,
\end{aligned} \tag{5.14}$$

$$\begin{aligned}
cw_{tx} + c\,dw_{xx} + bw_{txx} + (c + b\,d)w_{xxx} \\
+aw_{txxx} + (b + a\,d)w_{xxxx} + (a + cv)w_{xxxxx} \\
+bvw_{xxxxxx} + avw_{xxxxxxx} = 0.
\end{aligned} \tag{5.15}$$

Each of the equations (5.10)–(5.15) is a homogeneous function in the partial derivatives of $w(x,t)$. Therefore, we can assume a solution in the form

$$w(x,t) = 1 + e^{\alpha x + \beta t + \gamma}, \tag{5.16}$$

where α, β, and γ are to be determined later. Substituting for w in (5.10) gives

$$-4\alpha^6 bv = 0, \tag{5.17}$$

which yields $b = 0$. Inserting (5.16) and the value of b into (5.11)–(5.15), we obtain, respectively,

$$a - 19cv = 0, \tag{5.18}$$

$$a\beta + a\alpha\, d + \alpha^2 (10a - 160cv) = 0, \tag{5.19}$$

$$6a\beta + c + 6a\alpha\, d + a\alpha^4 v - \frac{30c^2 v}{a} + \alpha^2 (25a - 305cv) = 0, \tag{5.20}$$

$$\beta c + c\, d\alpha + \left(7a\beta + 3c - \frac{60}{a} c^2 v\right) \alpha^2 + 7a\, d\alpha^3 + (15a - 105cv)\, \alpha^4 + 3av\alpha^6 = 0, \tag{5.21}$$

$$\left(a\alpha^2 + c\right) \left(\alpha^2 + \beta + \alpha\, d + \alpha^4 v\right) = 0. \tag{5.22}$$

The solution of the system of algebraic equations given by (5.18)–(5.22) gives the following values of the parameters involved:

$$\begin{aligned} c &= \frac{a}{19v}, \\ \alpha &= \pm\sqrt{\frac{11}{19v}}, \\ \beta &= -\sqrt{\frac{11}{6859v}} \left(19\, d \pm 30\sqrt{\frac{11}{19v}}\right), \end{aligned} \tag{5.23}$$

and a and d are arbitrary.

Since $w(x,t)$ defined in (5.16) satisfies (5.10)–(5.15) and $f(w)$ given by (5.7) satisfies (5.6), then $u(x,t)$ given by (5.4) is actually a solution to (5.1). From (5.4) and (5.7) we obtain the following new solution of the KS equation:

$$u = -\frac{30}{19}\sqrt{\frac{11}{19v}} \left(\tanh\frac{\phi}{2} + 1\right) \left[\frac{11}{2} \left(\tanh\frac{\phi}{2} + 1\right)^2 \mp 1\right] + d, \tag{5.24}$$

where

$$\phi = \alpha x + \beta t + \gamma \tag{5.25}$$

and γ is arbitrary. By setting each of the constants d and γ to zero and v to unity, the solution (5.24) reduces to the solution obtained using the Backlund transformation [16] and to the solution obtained using Weiss-Tabor-Carnevale method [17].

6. CONCLUSION

By balancing the nonlinear and dispersive effects in a nonlinear partial differential equation, a transformation which reduces the nonlinear equation is obtained. The presented approach makes use of the computational power offered by symbolic manipulators for finding exact solutions of different nonlinear evolution equations. The significance of this approach is that it can be used to search for exact solution of other nonlinear equations.

REFERENCES

1. G. Bluman and S. Kumei, On the remarkable nonlinear diffusion equation $\frac{\partial}{\partial x}[a(u+b)^{-2}\frac{\partial u}{\partial x}] - \frac{\partial u}{\partial t} = 0$, *J. Math. Phys.* **21** (5), 1019–1023, (1980).
2. J.D. Cole, On a quasilinear parabolic equations occurring in aerodynamics, *Quart. Appl. Math.* **9**, 225–236, (1951).
3. E. Hopf, The partial differential equation $u_t + uu_x = \mu u_{xx}$, *Comm. Pure Appl. Math.* **3**, 201–230, (1950).
4. S. Kumei and G. Bluman, When nonlinear differential equations are equivalent to linear differential equations, *SIAM J. Appl. Math.* **42** (5), 1157–1173, (1982).
5. M. Torrisi, R. Tracina and A. Valenti, A group analysis approach for a nonlinear differential system arising in diffusion phenomena, *J. Math. Phys.* **37** (9), 4758–4767, (1996).

6. Y. Gao and B. Tian, Using symbolic computation to exactly solve for the Bogoyavlenskii's generalized breaking soliton equation, *Computers Math. Applic.* **33** (4), 35–37, (1997).
7. Y. Gao and B. Tian, Generalized tanh method with symbolic computation and generalized shallow water wave equation, *Computers Math. Applic.* **33** (4), 115–118, (1997).
8. A.V. Manickam, K.M. Moudgalya and A.K. Pani, Second-order splitting combined with orthogonal cubic spline collocation method for the Kuramoto-Sivashinsky equation, *Computers Math. Applic.* **35** (6), 5–25, (1998).
9. J.M. Burgers, *A Mathematical Model Illustrating the Theory of Turbulence*, Academic Press, New York, (1948).
10. M.J. Lighthill, Viscosity effects in sound waves of finite amplitude, In *Surveys in Mechanics*, (Edited by G.K. Batchelor and R.M. Davies), Cambridge Univ. Press, London, (1959).
11. D.T. Blackstock, Thermoviscous attenuation of plane, periodic, finite amplitude sound waves, *J. Acoust. Soc. Amer.* **36**, 534–542, (1964).
12. G.B. Whitham, *Linear and Nonlinear Waves*, Wiley, New York, (1974).
13. B. Nicolaenko, B. Scheurer and R. Temam, Some global dynamical properties of the Kuramoto-Sivashinsky equation: Nonlinear stability and attractors, *Physica D* **16**, 155–183, (1985).
14. R.D. Russell, D.M. Sloan and M.R. Trummer, Some numerical aspects of computing inertia manifolds, *SIAM J. Sci. Comp.* **14**, 19–43, (1993).
15. B. Garcia-Archilla, Some practical experience with the time integration of dissipative equations, *J. Comp. Phy.* **122**, 25–29, (1995).
16. N.A. Kudryashov, Exact soliton solution of the generalized evolution equation of wave dynamics, *J. Appl. Math. and Mech.* **52**, 361–365, (1988).
17. N.A. Kudryashov, Exact solution of the generalized Kuramoto-Sivashinsky equation, *Physics Letters A* **147** (5/6), 287–291, (1990).